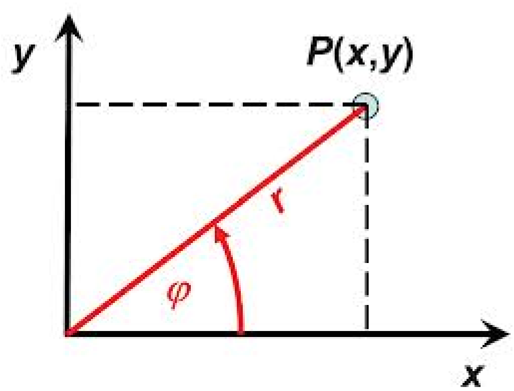
Phys 410
Fall 2014
Lecture #2 Summary
4 September, 2013

We considered Newton's second law of motion in two-dimensional polar coordinates (r, φ): $m\ddot{\vec{r}} = F_{net}$. The basic issue is that the polar unit vectors $\hat{r}$ and $\hat{\varphi}$ change direction as the particle moves about the plane. Note that $\hat{r}$ is defined as the direction of increasing r coordinate at fixed φ coordinate, and similarly for $\hat{\varphi}$. Both unit vectors are functions of the angular coordinate φ. They can be written in terms of the directionally-invariant Cartesian unit vectors $\hat{\imath}$ and $\hat{\jmath}$ as $\hat{r} = cos\varphi\,\hat{\imath} + sin\varphi\hat{\jmath}$, and $\hat{\varphi} = -sin\varphi\,\hat{\imath} + cos\varphi\hat{\jmath}$. A moving particle is parameterized by the time-dependent functions $r(t)$ and $\varphi(t)$, yielding a time-varying pair of unit vectors $d\hat{r}/dt = \dot{\varphi}\,\hat{\varphi}$ and $\frac{d\hat{\varphi}}{dt} = -\dot{\varphi}\,\hat{r}$. Starting from the coordinate vector $\vec{r} = r\hat{r}$, one can take two derivatives to find the rather complicated result $\vec{a} = (\ddot{r} - r\dot{\varphi}^2)\hat{r} + (r\ddot{\varphi} + 2\dot{r}\dot{\varphi})\hat{\varphi}$.



We considered projectile motion with air resistance. The drag force is directed opposite to the instantaneous velocity and depends on the speed as $\vec{f} = -f(v)\hat{v}$, where $f(v) = 0 + bv + cv^2$, and $\hat{v} = \frac{\vec{v}}{|\vec{v}|} = \frac{\vec{v}}{v}$ is the direction unit vector of the velocity. The linear term bv in $f(v)$ arises from viscous friction as the object moves through the fluid (air, water, honey, etc.), and depends on the linear dimension of the object, *D*. The quadratic term cv^2arises from accelerating the fluid out of the way of the object, and depends on the cross-sectional area of the object, D^2. The ratio of the two forces is $\frac{f_{quad}}{f_{lin}} = 1.56 \times 10^3\,\frac{s}{m^2}Dv$ for a spherical object of diameter *D* moving through air at STP (20^oC, 1 atmosphere pressure). This ratio is directly related to the Reynold's number in hydrodynamics. We found that the quadratic force dominates the linear force for a free-falling skydiver, and a golf ball flying through the air. The two forces are comparable for a falling rain-drop, and the linear force dominates for a small oil particle falling in air in the Millikan oil drop experiment.

An object falling under the influence of gravity and the linear drag force obeys the equation of motion: $m\dot{\vec{v}} = m\vec{g} - b\vec{v}$. In a Cartesian coordinate system where x is horizontal and y points downward, the vector equation breaks cleanly in to two un-coupled scalar first-order equations: $m\dot{v}_x = 0 - bv_x$, and $m\dot{v}_y = mg - bv_y$. The horizontal and vertical equations can be solved separately, and the full solution can be constructed later.

The horizontal equation can be written as $\dot{v}_x = -\frac{b}{m}v_x = -\frac{1}{\tau}v_x$, where we have defined a characteristic time of $\tau \equiv m/b$. Integrating the equation after separating variables, one finds $v_x(t) = v_{x0}e^{-t/\tau}$, where $v_x(0) = v_{x0}$ is the initial velocity. The velocity simply decays to zero after its initial kick, and the "1/e decay time" is τ. The velocity equation can be integrated to find the position of the particle moving horizontally under the influence of linear drag: $x(t) - x(0) = x_\infty\left(1 - e^{-t/\tau}\right)$, where $x_\infty = v_{x0}\tau$. The particle only moves a finite distance x_∞ before effectively coming to rest.

We then considered vertical motion of a point-like object subjected to a linear drag force. Newton's second law of motion for the vertical component of motion is given by $m\dot{v}_y = mg - bv_y$, with y-positive in the downward direction. Starting from rest, the particle will initially experience no drag force. As it accelerates downward, the drag force will increase steadily. At some point the drag force will equal the force of gravity, and the acceleration will cease. The object will be moving at the terminal velocity from that time onward, given by the balance of the two forces as $v_{ter} = \frac{gm}{b} = \tau g$. The equation of motion can be written as $m\dot{v}_y = b\left(v_y - v_{ter}\right)$, with a solution $v_y(t) = v_{y0}e^{-t/\tau} + v_{ter}\left(1 - e^{-t/\tau}\right)$. This equation shows that the velocity starts as v_{y0} and then switches over to v_{ter} as time evolves – in some sense it eventually 'forgets' the initial velocity and adopts the terminal velocity. The time scale for this crossover is $\tau = m/b$, the characteristic time governing the horizontal motion as well. The velocity equation can be integrated to find the vertical position as a function of time: $y(t) - y(0) = v_{ter}t - \left(v_{y0} - v_{ter}\right)\tau\left(1 - e^{-t/\tau}\right)$.

Combining the solutions for the x-motion and y-motion, we can now construct a solution (trajectory) for general motion in the xy-plane for a particle experiencing linear drag. Solving the $x(t)$ equation from above for t in terms of x, and putting this into the $y(t)$ equation above, we can find the trajectory of the particle: $y(x) = \frac{v_{y0} + v_{ter}}{v_{x0}}x + v_{ter}\tau \ln\left(1 - \frac{x}{v_{x0}\,\tau}\right)$ (now it is assumed that positive y is in the "up" direction). The trajectory in the absence of drag is given by $y(x) = \frac{v_{y0}}{v_{x0}}x - \frac{1}{2}g\left(\frac{x}{v_{x0}}\right)^2$. The main difference is in the second term. The logarithm is negative for all non-zero values of x, and has a negative divergence as the argument goes to zero. This occurs when $x = \tau\, v_{x0}$, showing that the horizontal range of the projectile is limited by this value. Hence when Felix Baumgartner stepped out of the capsule and gave himself an initial horizontal velocity (v_{x0}), his net displacement in the horizontal direction was limited (assuming that a linear drag force acted upon him at least in the initial part of the fall).

We considered motion with quadratic drag, $f = -c\,v^2$. The equation of motion is $m\dot{\vec{v}} = m\vec{g} - c\,v^2\hat{v}$. Note that the last term can also be written as $-c\,v\,\vec{v}$, allowing a decomposition into two scalar differential equations: $m\dot{v}_x = 0 - c\,\sqrt{v_x^2 + v_y^2}\,v_x$ and $m\dot{v}_y = mg - c\,\sqrt{v_x^2 + v_y^2}\,v_y$. Note that these equations do not separate cleanly into a v_x-only and a v_y-only set of equations, as they did in the linear drag case. We will attack coupled equations like this in the next lecture. The equations are also nonlinear. In this case there is no analytical general solution for this pair of equations (nonlinear equations will also be addressed later in the semester). For now we will consider motion exclusively in the x-direction to simplify the problem.

If we confine the particle to move only in the the x-direction ($v_y = 0$), the first equation reduces to $m\dot{v}_x = -c\,v_x^2$, with solution $v_x = \frac{v_0}{1+t/\tau}$, where we have defined a new characteristic time scale $\tau \equiv m/v_0 c$, and v_0 is the initial velocity. Here we see that the velocity relaxes more slowly than in the linear drag case, where the relaxation was exponential rather than algebraic. The velocity equation can be integrated to find the position of the particle along the x-axis: $x(t) - x(0) = v_0\tau\ln(1 + t/\tau)$. In this case the particle continues to move forever in the x-direction as time increases. Solving the equation for the vertical-only motion is left as a homework problem.